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MATHEMATICAL ANALYSIS

(INTEGRATED CALCULATION AND NUMERICAL SERIES)

Lecture notes

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MATHEMATICAL ANALYSIS

(INTEGRATED CALCULATION AND NUMERICAL SERIES)

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CONTENT

P.

INTEGRATED CALCULATION OF THE FUNCTION OF ONE VARIABLE

1. Initial function and indefinite integral

Definition. The function $F(x)$ is called the initial function of $f(x)$ over the interval [a; b] if $F(x)$ is a differentiable function on $[a; b]$ and F$F'(x) = f(x), \quad x \in [a; b]$.

Theorem. If $F_1(x)$ and $F_2(x)$ are antiderivatives of $f(x)$ in same interval $[a; b]$, the difference between them is equal to a constant, called the constant of integration.

Definition. If $F(x)$ is an antiderivatives of the initial function $f(x)$ over the interval $[a; b]$ and $C = const$, then the expression $F(x) + C$ is called the ***indefinite integral*** of the function $f(x)$ on $[a; b]$ and is denoted by the integral symbol $\int f(x)dx$. Thus, the integral symbol $\int f(x)dx$ means each of many antiderivatives of a function $f(x)$: $\int f(x)dx = F(x) + C$, where the symbol $\int$ is called an integral; the function $f(x)$ – the integral function, or the integrand $f(x)\,dx$ – an integral expression, x – an integral variable.

Definition. The operation of finding the indefinite integral of a function is called the integration of this function.

From the point of view of geometry, the indefinite integral is a set of curves, each of which is called an integral curve and is formed by shifting one of them parallel to itself along the axis O_y. To extract a certain integral curve $F(x)$ from this set, it suffices to set its value $F_0(x)$ at some point $x_0 \in [a; b]$.

2. Properties of Indefinite Integrals

1. The derivative of the indefinite integral is equal to the integrand function: $(\int f(x)dx)' = f(x)$.

Indeed, if $F'(x) = f(x)$, then

$$(\int f(x)dx)' = (F(x) + C)' = F'(x) = f(x).$$

2. The differential from the indefinite integral is equal to the integrand expression: $d(\int f(x)dx) = f(x)dx$.

Indeed,

$$d \int f(x)dx = (\int f(x)dx)'dx = f(x)dx.$$

3. The indefinite integral of the differential of a function is equal to the sum of this function and an arbitrary constant:

$$\int d\, F(x) = F(x) + C.$$

4. A constant factor can be taken as the sign of the indefinite integral: $\int C\, f(x)dx = C \int f(x)dx$.

5. The indefinite integral of the algebraic sum of two functions is equal to the algebraic sum of the integrals of these functions:

$$\int f(x) \pm g(x)dx = \int f(x)dx \pm \int g(x)dx.$$

Property 5 is valid for an arbitrary finite number of terms.

6. If $\int f(x) = F(x) + C$ and $u = \varphi(x)$ – an arbitrary function having a continuous derivative, then $\int f(u) = F(u) + C$.

Property 6 (it is called the invariance of the integration formula) is very important. It means that one or another formula for an indefinite integral remains valid regardless of whether the integration variable is an independent variable or an arbitrary function of it with a continuous derivative. Thus, the number of calculated or, as they say, taken integrals increases indefinitely.

Theorem. Every continuous function on the segment [a; b] has a primary function on this segment.

In this regard, we will further assume that the integrable function is considered only on those segments where it is continuous.

3. Table of Integrals

Let $u\,(x)$ be an arbitrary function that has a continuous derivative $u'(x)$ on some interval. Then the following formulas are valid in this interval:

1. $\int du = u + C.$

2. $\int u^{\alpha} du = \frac{u^{\alpha+1}}{\alpha+1} + c, \ \ \alpha \neq -1.$

3. $\int \frac{du}{u} = \ln|u| + C.$

4. $\int e^{u} du = e^{u} + C.$

5. $\int a^{u} du = \frac{a^{u}}{\ln a} + c, \ \ a > 0, \alpha \neq 1.$

6. $\int \sin u \, du = -\cos u + C.$

7. $\int \cos u \, du = \sin u + C.$

8. $\int \frac{du}{\cos^2 u} = tg\, u + C.$

9. $\int \frac{du}{\sin^2 u} = -ctg\, u + C.$

10. $\int \frac{du}{1+u^2} = arctg\, u + C.$

11. $\int \frac{du}{\sqrt{1-u^2}} = arcsin\, u + C.$t

12. $\int \frac{du}{a^2+u^2} = \frac{1}{a} arctg\, \frac{u}{a} + C.$

13. $\int \frac{du}{\sqrt{a^2-u^2}} = arcsin\ \frac{u}{a} + C.$

14. $\int tg\, u\, du = -\ln|\cos u| + C.$

15. $\int ctg\, u\, du = \ln|\sin u| + C.$

16. $\int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C.$

17. $\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right| + C.$

18. $\int \frac{du}{\sqrt{u^2+a^2}} = \ln\left|u + \sqrt{u^2 + a^2}\right| + C.$

19. $\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u + \sqrt{u^2 - a^2}\right| + C.$

20. $\int \frac{du}{\sin u} = \ln\left|tg\ \frac{u}{2}\right| + C.$

21. $\int \frac{du}{\cos u} = \ln\left|tg\ \left(\frac{u}{2} + \frac{\pi}{4}\right)\right| + C.$

22. $\int sh\, u\, du = ch\, u + C.$

23. $\int ch\, u\, du = sh\, u + C.$

24. $\int \frac{du}{ch^2 u} = th\, u + C.$

25. $\int \frac{du}{sh^2 u} = -cth\, u + C.$

26. $\int th\, u\, du = \ln|ch\, u| + C.$

27. $\int cth\, u\, du = \ln|sh\, u| + C.$

Integrals 1 – 27 are called tabular integrals. Part of these formulas follows directly from the definition of the indefinite integral, the table of derivatives and property 6 of the indefinite integral.

The validity of other formulas is easy to check by differentiation.

4. Basic Methods of Integration

The main methods of integration are direct integration, the method of substitution (variable replacement) and the method of integration by parts.

1. The ***method of direct integration*** is the calculation of integrals using the basic properties of the indefinite integral and the table of integrals.

2. The ***substitution method (variable replacement)*** is to introduce a new integration variable.

Suppose we need to find the integral $\int f(x)dx$, and we cannot directly pick up the original for $f(x)$, but we know that it exists.

We introduce the replacement of a variable: $x = \varphi(t)$, where $\varphi(t)$ – a continuously differential function for which there is an inverse function. Then the equality is true:

$$\int f(x)dx = f(\varphi(t) \cdot \varphi'(t)dt \qquad (1)$$

(here it is implied that after integration in the right part of equality (1) instead of the variable t we will substitute its expression through x, found from replacement $x = \varphi(t)$).

3. Now, consider the ***method of integration by parts***.

Let functions $u = u(x)$ and $v = v(x)$ have continuous derivatives on some interval. Then

$$d(u \cdot v) = u \cdot dv + v \cdot du.$$

By integrating both parts, we have

$$u \cdot v = \int u \cdot dv + \int v \cdot du$$

or

$$\int u\, dv = u \cdot v - \int v du. \qquad (2)$$

Formula (2) is called the formula for integrating parts.

DEFINITE INTEGRAL

5. Definite Integral. Conditions for Existence of Definite Integrals

Definition 1. The number I is called the limit of the integral sum $\sigma = \sum_{k=0}^{n-1} f(C_k)\Delta x_k$ for $\lambda(T) \to 0$ if for any $\varepsilon > 0$, there exists a number $\delta > 0$ that as soon as $\lambda(T) < \delta$, then for any selection of points C_k and any T – partition of the segment $[a; b]$, the inequality $|\sigma - i| < \varepsilon$ is true.

In fact the number I is the limit of the integral sum σ and is written as follows:

$$I = \lim_{\lambda(T)\to 0} \sum_{k=0}^{n-1} f(C_k)\Delta x_k. \qquad (3)$$

Definition 2. The boundary of an integral sum, if it exists, is called the definite integral of the function $f(x)$ on the segment $[a; b]$ and is denoted

$$I = \int_a^b f(x)dx. \qquad (4)$$

The number a is called the lower limit of integration, b – the upper limit; $f(x)$ is called an integrable function; $f(x)\,dx$ – integrable expression; segment $[a; b]$ – the interval of integration.

If the boundary of the integral sum, or the definite integral of the function $y = f(x)$, exists, then such a function is called the function integrated on the segment $[a; b]$.

Theorem. Any function continuous on the segment $[a; b]$ integrates on this segment.

6. Properties of Definite Integrals

We formulate and prove the properties of a definite integral for a continuous function.

1. Let the function $f(x)$ be given and continuous on the interval $[a; b]$, $a < b$. Then there is a definite integral

$$\int_a^b f(x)dx$$

and the equality

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

is true.

2. For any function $y = f(x)$,

$$\int_a^b f(x)dx = 0.$$

3. There is an integral $\int_a^b f(x)dx$, where C is an arbitrary constant, and the equality

$$\int_a^b Cf(x)dx = C\int_a^b f(x)dx$$

is true.

Property 3 is formulated as follows: A constant factor can be taken as the sign of a definite integral.

4. On the segment $[a; b]$ continuous functions $f(x)$ and $\varphi(x)$ are given. Then there is an integral

$$\int_a^b (f(x) + \varphi(x))dx$$

and the equality

$$\int_a^b (f(x) + \varphi(x))dx = \int_a^b f(x)dx + \int_a^b \varphi(x)dx$$

is satisfied.

Property 4 is formulated as follows: The definite integral of the sum of a function is equal to the sum of the definite integrals of these functions.

IMPROPER INTEGRALS

7. Improper Integrals with Infinite Boundaries

Earlier, we considered a definite integral on a finite segment [a; b]. However, in a number of problems it is necessary to consider the integral at infinite intervals [a; $+\infty$], [$-\infty$; b], [$-\infty$; $+\infty$]. It is clear that the notion of a definite integral cannot be directly applied to these cases. You can't even plot an integral sum. Therefore, it is necessary to look for other methods and to enter new definitions of the definite integral in each of these cases.

Therefore, let the function $f(x)$ be defined, for example, on the interval [a; $+\infty$], being continuous on any segment [a; b] where $b > a$ is an arbitrary real number. Then there is a definite integral

$$\int_a^b f(x)dx$$

and it is a function of the upper limit (a is a constant number):

$$F(b) = \int_a^b f(x)dx.$$

Definition 1. If there is a finite integral boundary $\int_a^b f(x)dx$ for $b \to +\infty$, then this boundary is called the integral $f(x)$ from a to $+\infty$ and is written as

$$\int_a^{+\infty} f(x)dx = \lim_{b\to+\infty} \int_0^b f(x)dx. \qquad (4)$$

In this case, the integral $\int_0^{+\infty} f(x)dx$ is called convergent, and the function $f(x)$ itself is called integrated on the interval $[a; +\infty]$. If the boundary (4) is an improper number ($+\infty$ or ∞) or does not exist at all, then the integral $\int_a^{\infty} f(x)dx$ is called divergent.

The improper integral is similarly denoted as $\int_{-\infty}^b f(x)dx$.

Let the function $f(x)$ be defined on the interval $[-\infty; b]$ and continuous on any segment $[a; b]$, where a is an arbitrary real number and $a < b$. Then the definite integral $\int_a^b f(x)dx$ is a function of the lower bound $\Phi(a) = \int_a^b f(x)dx$.

Definition 2. If there is a finite boundary of the integral $\int_a^b f(x)dx$ (function $\Phi(a)$) for $a \to -\infty$, then this boundary is called the integral of the function $f(x)$ from $-\infty$ to a and is written as

$$\int_{-\infty}^b f(x)dx = \lim_{a\to+\infty} \int_a^b f(x)dx. \qquad (5)$$

There, the integral $\int_{-\infty}^{b} f(x)dx$ is called convergent, and the function $f(x)$ itself is called integrable in the interval [–∞; b]. If the boundary (5) is an eigenvalue or does not exist at all, then the integral $\int_{-\infty}^{b} f(x)dx$ is called divergent. The integral $\int_{-\infty}^{b} f(x)dx$ as well as the integral $\int_{a}^{+\infty} f(x)dx$ is called improper.

If the function $f(x)$ is defined on the interval [a; b], where a and b are arbitrary real numbers, then we can denote the integral from – ∞ to + ∞, namely,

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{+\infty} f(x)dx, \qquad (6)$$

where c is an arbitrary number. The integral $\int_{-\infty}^{+\infty} f(x)dx$ is called improper. In this case, if the integrals in the right-hand side of equation (6) coincide, then the improper integral $\int_{-\infty}^{+\infty} f(x)dx$ is called convergent. If at least one of the integrals of the right-hand side of equation (6) diverges, then the improper integral $\int_{-\infty}^{+\infty} f(x)dx$ is called divergent.

8. Double Integral and Conditions for Their Existence

Let the function $z = f(x, y)$ be defined in a closed square bounded domain $\overline{D} \subset R_2$. Divide the area $\overline{D}$ by a grid of curves into n arbitrary parts $\overline{D}_k$, where $k = 0, 1, 2, \dots, n-1$, in pairs without common internal points so that $\overline{D} = \bigcup_{k=1}^{n-1} \overline{D}_k$. Let's call this partition T partition of domain $\overline{D}$. The areas of the parts $\overline{D}_k$ are denoted by ΔS_k, where $k = 0, 1, 2, \dots, n-1$. In each part $\overline{D}_k$ we choose a point $P_k(x_k, y_k)$, $P_k \in \overline{D}_k$, $k = 0, 1, 2, \dots, n-1$, and form the sum

$$\sigma = \sum_{k=0}^{n-1} f(x_k; y_k)\Delta S_k. \qquad (7)$$

This sum is called the double integral sum for the function $z = f(x, y)$ in the domain $\overline{D}$.

Let $d_k = diam\overline{D}_k, k = 1,2,\dots,n-1$, then

$$\lambda(T) = \max_{0 \le k \le n-1} d_k$$

Definition 1. The number I is called the limit of integral sums σ for $\lambda(T) \to 0$ if, for an arbitrary number $\varepsilon > 0$, there exists a number $\delta(\varepsilon) > 0$ such that the inequality $|\sigma - I| < \varepsilon$ is satisfied by an arbitrary T – partition of the domain $\overline{D}$, for which $\overline{D} = \bigcup_{k=1}^{n-1} \overline{D}_k$ and any choice of points $P_k(x_k, y_k)$, where $P_k \in \overline{D}_k$ and $k = 0, 1, 2, \dots, n-1$ as soon as $\lambda(T) < \sigma$.

Definition 2. If for $\lambda(T) \to 0$ the integral sums σ have the boundary number I, then this number is called the double integral of the function $f(x; y)$ over the domain $\overline{D}$ and is denoted by

$$\iint f(x;y)dS \;\; or \;\; \iint f(x;y)dxdy.$$

Therefore, by definition,

$$\iint f(x;y)dxdy \;\; = \lim_{\lambda(T)\to 0} \sum_{k=0}^{n-1} f(x_k;y_k)\Delta S_k, \qquad (8)$$

if the boundary (8) exists, i.e. there is a double integral of the function $f(x;y)$ over the domain $\overline{D}$, then the function $f(x;y)$ is called integrable (according to Riemann) in the domain $\overline{D}$.

9. Properties of Double Integrals

1. If $f(x;y) = C$, $C - const$, $(x, y) \in \overline{D}$, then

$$\iint Cdxdy \;\; = CS, \qquad (9)$$

where S is the area of the domain $\overline{D}$.

2. If the functions $f(x;y)$ and $\varphi(x;y)$ are integrable on the domain $\overline{D}$, then the functions $f(x;y) \pm \varphi(x;y)$ are also integrable on this domain and the equality holds:

$$\iint f\,(x;\; y) \pm \varphi\,(x;\; y)dxdy = \iint f\,(x;\; y)dxdy \pm$$

$$\pm \iint \varphi\,(x;\; y)dxdy.$$

3. If the function $f(x; y)$ is integrable on the domain $\overline{D}$, then the function $C f(x; y)$ is also integrable on this domain, where C is *const*, and

$$\iint Cf\,(x;\ y)dxdy = C\iint f\,(x;\ y)dxdy.$$

4. If $f(x; y) \geq 0$, $(x, y) \in \overline{D}$ and the function $f(x; y)$ is integrable on the domain $\overline{D}$, then $\iint f\,(x;\ y)dxdy \geq 0$.
5. If $f(x; y) \geq \varphi(x; y)$, $(x, y) \in \overline{D}$ and each of the functions $f(x; y)$ and $\varphi(x; y)$ is integrated in the domain $\overline{D}$, then $\iint f\,(x;\ y)dxdy \geq \iint \varphi\,(x;\ y)dxdy$.

10. The Concept of Triple Integrals and Conditions for Their Existence

Let the function $u = f\,(x,\ y, z)$ be defined in a closed square bounded domain $\bar{G} \subset R_3$. Divide the domain $\bar{G}$ by a grid of curves into n arbitrary parts $\bar{G}_k$, where $k = 0, 1, 2, \ldots, n-1$, in pairs without common internal points so that $\bar{G} = \bigcup_{k=1}^{n-1} \bar{G}_k$. Let's call this partition T partition of domain $\bar{G}$. The areas of the parts $\bar{G}_k$ are denoted by ΔV_k, $k = 0, 1, 2, \ldots, n-1$. In each region $\bar{G}_k$, we choose a point $P_k(x_k, y_k, z_k)$, where $V_k \in \bar{G}_k$ and $k = 0, 1, 2, \ldots, n-1$, and form the sum

$$\sigma = \sum_{k=0}^{n-1} f(x_k; y_k, z_k)\Delta V_k. \qquad (10)$$

This sum is called the double integral sum for the function $z = f(x, y, z)$ in the domain $\bar{G}$.

Let $d_k = diam\bar{G}_k, k = 1,2, \dots, n-1$.

$$\lambda(T) = \max_{0 \le k \le n-1} d_k.$$

Definition 1. The number I is called the limit of integral sums σ for $\lambda(T) \to 0$ if for an arbitrary number $\varepsilon > 0$, there exists a number $\delta(\varepsilon) > 0$ such that the inequality $|\sigma - I| < \varepsilon$ is satisfied by an arbitrary T partition of the domain $\bar{G}$, $\bar{G} = \bigcup_{k=1}^{n-1} \bar{G}_k$ and any choice of points $P_k(x_k, y_k, zy_k)$, $P_k \in \bar{G}_k$, $k = 0, 1, 2, \dots, n-1$ as soon as $\lambda(T) < \sigma$.

Definition 2. If for $\lambda(T) \to 0$ the integral sums σ have the boundary number I, then this number is called the double integral of the function $f(x; y)$ over the domain $\bar{G}$ and is denote by

$$\iiint f(x; y; z)dV \; or \; \iiint f(x; y; z)dxdydz.$$

Thus, by definition,

$$\iiint f(x; y; z)dxdydz = \lim_{\lambda(T)\to 0} \sum_{k=0}^{n-1} f(x_k; y_k; z_k)\Delta V_k, \quad (11)$$

if the boundary (10) exists, i.e. there is a double integral of the function $f(x; y; z)$ over the domain $\bar{G}$, then the function $f(x; y; z)$ is called integrable (according to Riemann) on the domain $\bar{G}$:

$$m = \iiint \gamma f(x; y; z) dxdydz. \tag{12}$$

11. Properties of Triple Integrals

If $f(x; y; z) = C$, C – *const*, $(x, y, z) \in \bar{G}$, then

$$\iiint C dxdydz = CV, \tag{13}$$

where V is the area of the domain $\bar{G}$. For $C = 1$ and $C = 0$ from equation (13), in particular, we obtain $\iiint dxdydz = V$ and $\iiint 0 dxdydz = 0$, respectively.

2. If the functions $f(x; y; z)$ and $\varphi(x; y; z)$ are integrable on the domain $\bar{G}$, then the functions $f(x; y; z) \pm \varphi(x; y; z)$ are also integrable on this domain and the equality holds:

$$\iiint f(x; y; z) \pm \varphi(x; y; z) dxdydz =$$

$$= \iiint f(x;\ y; z) dxdydz \pm \iiint \varphi(x;\ y) dxdy.$$

3. If the function $f(x; y; z)$ is integrable on the domain $\bar{G}$, then the function $C f(x; y; z)$ is also integrable, on this domain where C is *const*, and

$$\iiint Cf\,(x;\ y; z)dxdydz = C\iiint f\,(x;\ y; z)dxdydz.$$

4. If $f(x; y; z) \geq 0$, $(x, y; z) \in \bar{G}$ and the function $f(x; y; z)$ is integrable on the domain $\bar{G}$, then $\iiint f\,(x;\ y; z)dxdydz \geq 0$.

5. If $f(x; y; z) \geq \varphi\,(x; y; z)$, $(x, y, z) \in \bar{G}$ and each of the functions $f(x; y; z)$ and $\varphi\,(x; y; z)$ is integrable on the domain $\bar{G}$, then $\iiint f\,(x;\ y; z)dxdydz \geq \iiint \varphi\,(x;\ y; z)dxdydz$.

NUMERICAL SERIES

12. General Definitions of the Theory of Series

Definition. Given is a sequence of numbers added together:

$$u_1 + u_2 + \cdots + u_n + \cdots. \tag{14}$$

The numbers u_1, u_2, …, u_n, … are called the terms series.

When the sequence are added together, the sum of the terms is called series.

The sum of a finite number n of the first terms of the series is called the partial sum of the series and is denoted as follows:

$$S_n = u_1 + u_2 + \cdots + u_n. \tag{15}$$

When n changes, so does S_n.

Therefore, each series corresponds to a sequence of its partial sums S_1, S_2, S_3… On the contrary, each sequence S_1, S_2, S_3…, corresponds to a number:

$$S_1 + (S_2 - S_1) + (S_3 - S_2) + \cdots + (S_n - S_{n-1}) + \cdots.$$

A series is said to be convergent if there is a finite boundary in the sequence of its partial sums $\lim_{n\to\infty} S_n = S$. This boundary S is called the sum of the series. A series is called divergent if such a boundary does not exist.

Between the divergent series sometimes really diverging ones are distinguished, for which $S_n \to \infty$, i.e. $\lim_{n\to\infty} \frac{1}{S_n} = 0$.

If the series coincides, then write:

$$u_1 + u_2 + \cdots + u_n + \cdots + u_{n+1} + \cdots = S.$$

If we separate from the convergent series n its first terms, i.e. S_n, then the set of other terms $S - S_n = u_{n+1} + u_{n+2} + \ldots = r_n$ is called the remainder of the series.

13. Progressions

Arithmetic and geometric progressions are considered in elementary mathematics.

Member terms of the arithmetic progression form a sequence:

$$a, \qquad a + d, \qquad a + 2d, \ldots, \qquad a + (n-1)d, \ldots$$

If we connect the terms of this sequence with addition characters, we get a number:

$$a + (a + d) + (a + 2d) + \cdots + [a + (n-1)d] + \cdots$$

From elementary mathematics the formula for the partial sum of this series is known as:

$$S_n = \frac{a+a+(n-1)d}{2} \cdot n = an + \frac{n(n-1)}{2}d.$$

This part of the sum for $n \to \infty$ itself goes to infinity. Thus, a series whose terms are members of an arithmetic progression is actually divergent.

In elementary mathematics, as you know, only finite arithmetic progressions are considered.

Now, consider now a geometric progression. Its terms also form a sequence:

$$1, q, q^2, \dots, q^{n-1}, \dots$$

The terms of this sequence you can use to make a number:

$$1 + q + q^2 + \cdots + q^{n-1} + \dots$$

The partial sums of this series, as is known from elementary algebra, are written as follows:

$$S_n = \frac{q^n - 1}{q - 1}, if\ |q| > 1.$$

$$S_n = \frac{1 - q^n}{1 - q}, if\ |q| < 1.$$

We see that for $|q| > 1$ this series is actually divergent, and for $|q| < 1$ the series is convergent, and it is reasonable to talk about the sum of such a series. In the latter case, you can write:

$$\frac{1}{1-q} = 1 + q + q^2 + \cdots + q^{n-1} + \ldots$$

for $\lim_{n\to x} S_n = \lim_{n\to x} \frac{1-q^n}{1-q} = \lim_{n\to x} \left(\frac{1}{1-q} - \frac{q^n}{1-q}\right) = \frac{1}{1-q} - \frac{\lim_{n\to x} q^n}{1-q} =$

$= \frac{1}{1-q} = S.$

If $q = 1$, then $S_n = n$ and the series are actually divergent. If $q = -1$, then $S_n = 1 - 1 + 1 - 1 + \ldots + (-1)^{n-1}$ and for odd number n $S_n = 0$, and for even number n $S_n = 1$. Hence we see that there is no boundary $\lim_{n\to\infty} S_n$.

We see that geometric progression gives us samples of all types of series:

a) actually divergent for $|q| > 1$and $q = 1$;

b) divergent when $q = -1$ and

c) convergent when $|q| < 1$.

14. The Problem of Studying the Internal Convergence of a Series

In the theory of series, one of the main problems is the following: 1) establishing the very fact of convergency or divergency of the series under study and 2) establishing the amount of convergent series. It should be noted that the solution of both problems (especially of the second one) is very often reduced to a direct study of the process of changing partial sums of the series S_n for $n \to \infty$.

In the general case, finding the partial sums of the series S_n to determine the convergence or divergence of a given series is associated with significant difficulties, so we resort to other tests for convergence or divergence of the series, which we will prove below.

Next, we should focus on an important property of the series, namely:

Theorem 1. If the series obtained from the given series converges

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + \cdots + a_n + \cdots \qquad (16)$$

After discording some of its terms, then another side of the series (16) wice also converge. If this series coincides, then the series obtained by discarding some of the terms will also converge. Consequently, the convergence or divergence of the series is not violated if a certain finite number of terms of the series are discarded.

Here are two more simple properties of the series.

Theorem 2. If the series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots \qquad (17)$$

converges and its sum is equal to *S*, then the series

$$\sum_{n=1}^{\infty} ca_n = ca_1 + ca_2 + \cdots + ca_n + \cdots \qquad (18)$$

where *c* is an arbitrary fixed number, will also converge and its sum will be equal to *cS*.

Theorem 3. If the series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots \qquad (19)$$

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + \cdots + b_n + \cdots \qquad (20)$$

converge, and their sums are S' and S'', respectively, then the series

$$\sum_{n=1}^{\infty}(a_n + b_n) = (a_1 + b_1) + (a_2 + b_2) + \cdots + (a_n + b_n) + \cdots \qquad (21)$$

and

$$\sum_{n=1}^{\infty}(a_n - b_n) = (a_1 - b_1) + (a_2 - b_2) + \cdots + (a_n - b_n) + \cdots \qquad (22)$$

also converge and their sums are equal $S' + S''$ and $S' - S''$ respectively.

15. Tests for the Convergence of a Series

Let us consider several main necessary criteria for the convergence of a series.

Theorem. If the series

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + \cdots + u_n + \cdots$$

converges, then

$$\lim_{n\to\infty} u_n = 0 \tag{23}$$

If $\lim_{n\to\infty} u_n \neq 0$, then this series is probably divergent.

16. D'Alembert Ratio Test (for the Convergence of a Series)

Theorem. If in the series with positive terms the ratio of the next term to the previous $\frac{u_{n+1}}{u_n}$ starting from a certain value $n = n_0$ satisfies the inequality $\frac{u_{n+1}}{u_n} < q$, where the number q is constant and less than one, then the series probably converge. When, on the contrary, starting from a certain value $n = n_0$, we have $\frac{u_{n+1}}{u_n} \geq 1$, then this series probably diverges.

Conclusion. If there is $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$, then for $l < 1$ the series probably converges and for $l > 1$ the series probably diverges. In the case of $l = 1$, nothing definite can be said about the convergence or divergence of the series. This is a dubious case.

17. The Cauchy test (for the Convergence of a Series)

Theorem. If in a series with positive terms the common term, beginning with a certain value n, satisfies the inequality $\sqrt[n]{u_n} < q$, where the number q is a constant and less than one, then the series converges.

When, on the contrary, starting from a certain value n, we have $\sqrt[n]{u_n} \geq 1$, then the series diverges.

Conclusion. If there is $\lim\limits_{n\to\infty} \sqrt[n]{u_n} = l$, then at $l < 1$ the series probably converges, and at $l > 1$ the series probably diverges. The case $l = 1$ is doubtful here as well.

A more detailed analysis generally makes it easy to establish the fact that Cauchy test (criterion) is relatively “stronger” than D'Alembert ratio test, i.e. in all cases of applicability of D'Alembert test, Cauchy test is also applicable, but not vice versa.

In cases where D'Alembert ratio test is applicable, it is mostly advantageous in that it is simpler than Cauchy test, because the expression $\frac{u_{n+1}}{u_n}$ is often much simpler than the expression $\sqrt[n]{u_n}$.

18. The Integral Test for Convergence and Divergence of Series

Let us make some preliminary remarks on improper integrals with a positive integrant function.

Considering the improper integral $\int_a^\infty f\,(x)\,dx$, where $f\,(x) > 0$, we have, like in case of a series with positive terms, only two such possibilities: either the monotonically increasing value of the integral $I(b) = \int_a^b f\,(x)\,dx$ remains limited at $b \to +\infty$, then it has some finite boundary:

$$\lim_{b\to\infty}\int_a^b f\,(x)\,dx = \int_a^\infty f\,(x)\,dx$$

or $I\,(b) \to \infty$ for $b \to \infty$, and then the improper integral $\int_a^\infty f\,(x)\,dx$ does not exist, turning into infinity.

Theorem. If the function $f\,(x)$ is positive and monotonically decreasing at $x \geq a$ and goes to zero at $x \to +\infty$, then the series

$$f(a) + f(a+1) + f(a+2) + \cdots + f(a+n-1) +$$

$$+f(a+n)+f(a+n+1)+\cdots$$

converges if $\int_a^{\infty} f\,(x)\,dx$ exists, and diverges if this integral becomes infinity.

19. Alternating Series. The Convergence Test of Leibniz. Estimation of the remainder

Definition. Alternating series are called infinite series of the form

$$a_1-a_2+a_3-a_4+a_5-a_6+\cdots+a_n(-1)^{n-1}+\cdots,$$

where a_1, a_2, a_3… is a sequence of positive non-negative numbers.

Leibniz's theorem. If in the alternating series the absolute values of the common terms monotonically decrease to zero (i.e. $0 < a_1 > a_2 > a_3 > \ldots$, moreover $a_n = 0$ when $n \to \infty$), then the series converges, and its sum has a numerical value, intermediate between zero and the first term ($0 < S < a_1$).

Conclusion. By Leibniz's theorem, the remainder $S - S_n = r_n$ is smaller in absolute value than the first ommited term $|r_n| < a_{n+1}$ and has the same sign as this term.

20. Absolutely and Conditionally Convergent Series

Often the question of the convergence of a series whose members are positive and negative (such series are called alternating series), can be reduced to the question of the convergence of the positive series. Consider the following theorem:

Theorem. If a series $|u_1| + |u_2| + \ldots + |u_n| + \ldots$converges, then the series $u_1 + u_2 + \ldots + u_n + \ldots$ must also converge.

Definition. The convergent series $u_1 + u_2 + \ldots + u_n + \ldots$ is called absolutely convergent if the series $|u_1| + |u_2| + \ldots + |u_n| + \ldots$ also converges.

Definition. A series

$u_1 + u_2 + \ldots + u_n + \ldots$converges conditionally

if it converges, but

$$|u_1| + |u_2| + \ldots + |u_n| + \ldots \text{ diverges.}$$

We mentioned above that the notion of an absolutely convergent series makes it possible in some cases to determine the convergence of a series (if this series is absolutely convergent). But the meaning of this concept is far more important. The fact is that some properties of absolutely convergent series do not coincide with the corresponding properties of conditionally convergent series (which we will see below). Therefore quite often it is necessary to establish the fact of absolute or conditional convergence of a number even in the case when its convergence is already known to us.

Theorem. The sum of an absolutely convergent series remains unchanged at any permutation of its members.

Riemann's theorem. If a series coincides conditionally, then its members can be rearranged so that the newly formed series has any predetermined sum, or becomes divergent.

POWER SERIES

21. Radius and Interval of Convergence for a Series

A power series is a series of the form

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = \sum_{n=0}^{\infty} a_nx^n, \quad (24)$$

where x is a variable and $a_0, a_1, a_2, \ldots, a_n, \ldots$ are real numbers, called the coefficients of the series. The first thing to notice about a power series is that it is a function of x.

Power series are widely used in approximate calculations, which we will consider later. First of all, let us introduce the basic concepts concerning power series.

The power series (24) is convergent at the point $x = 0$, because at $x = 0$ it becomes a convergent numerical series

$$a_0 + a_10 + a_20 + \cdots + a_n0 = a_0$$

The region of convergence of a power series can be found in the same way as it was done for functional series.

The following theorem makes it possible to determine the interval of convergence for a power series.

Abel's theorem. If the power series (24)

$$\sum_{n=0}^{\infty} a_nx^n$$

convergent for $x = x_0 \neq 0$, then it is absolutely convergent for all values of x that satisfy the inequality $|x| < |x_0|$.

If for $x = x_0$ the power series is convergent, then it is divergent for all values of x for which $|x| > |x_0|$.

Definition. The interval of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ is such an interval $(-R; R)$ that for each point x, which lies within this interval, the series converges absolutely, and for points x, which are outside this interval, the series diverges. The number R is called the radius of convergence of the power series.

The question of the convergence of a series at $x = \pm R$ (at the endpoints of the interval) is solved for each series separately. Thus, the region of convergence of a power series may differ from its interval of convergence by no more than two points, which are the endpoints of the interval of convergence.

If the power series diverges everywhere except the point $x = 0$, then $R = 0$, and if the power series converges everywhere, then $R = +\infty$ and the interval of convergence of such a series is the whole numerical line – $(-\infty; +\infty)$.

22. Properties of Power Series

Theorem 1: about uniform convergence of power series segment.

The power series is uniformly convergent on each segment belonging to its interval of convergence.

Conclusion. The sum of a power series is continuous in the interval of its convergence.

T*heorem 2:* about the term wise differentiation of a power series.

If the power series (24) $\sum_{n=0}^{\infty} a_n x^n$ has the interval of convergence (– R; R), then the series

$$\sum_{n=0}^{\infty} n a_n x^{n-1}, \tag{25}$$

formed by the member differentiation of the series (24), has the same interval of convergence (– R; R). If $f(x)$ is the sum of the series (24) and $\varphi(x)$ is the sum of the series (25), then

$$\varphi(x) = f(x), x \in (-R; R). \tag{26}$$

Conclusion. The sum of the power series has derivatives of any order in the middle of the interval of convergence.

Theorem 3: about the term wise integration of a power series.

The power series can be articulated on each segment belonging to its interval of convergence.

23. Taylor's Series

It is known that the sum of a power series in the interval of its convergence is a continuous function differentiated an infinite number of times. Let us now consider under what conditions the given function $f(x)$ is the sum of the power series. Note that the problem of representing the function in the form of the power series is important because it is possible to approximate the function with the required accuracy by partial sums of the power series, which are polynomials. Then the calculation of the values of the function is reduced to the calculation of the values of the polynomial, i.e. to perform only the simplest arithmetic operations. Representation of the function in the form of the power series is used not only in calculating the values of the function but also in calculating integrals, solving equations, and so on.

Let the function $f(x)$ be defined around the point a and at this point have derivatives of any order. Assume that the function $f(x)$ can be represented as a power series in the interval $(a - R; a + R)$:

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + \cdots. \tag{27}$$

Express the coefficients a_0, a_1, …, a_n, … of the series (27) in terms of the values of the function $f(x)$ and its derivatives at the point a.

For $x = a$ from equation (27), we have $a_0 = f(a)$. According to the differentiation theorem for power series,

$$f'^{(x)} = a_1 + 2a_2(x-a) + \cdots + na_n(x-a)^{n-1} + \cdots$$

$$x \in (a-R; a+R). \tag{28}$$

In other words, if f has a power series expansion at a, then it must be of the form

$$\sum_{n=0}^{\infty} \frac{f_n(a)}{n!} (x-a)^n. \tag{29}$$

This series is called the Taylor series of function $f(x)$ about a (or centered at a).

Theorem 1. If the function $f(x)$ in the interval $(a - R; a + R)$ can be represented as the power series $\sum_{n=0}^{\infty} a_n(x-a)^n$, then this series is unique and it is a Taylor series of this function.

Theorem 2. For the Taylor series (29) to match the function $f(x)$ in the interval $(a - R; a + R)$, i.e. to hold

$f(x)=\sum_{n=0}^{\infty}\frac{f_n(a)}{n!}\,(x-a)^n, x\in(a-R;\ a+R,$ it is necessary and sufficient that in this interval the function $f(x)$ has derivatives of all orders and that the remainder of its Taylor formula goes to zero at $n\to\infty$ for all x in this interval.

Theorem 3. If the function $f(x)$ has derivatives of all orders the interval $(a-R;\ a+R)$ and there is such $M>0$ that

$$|f_n(x)|\leq M, where\ n=0,1,2,\ldots\ and\ x\in(a-R;\ a+R), \tag{30}$$

then the function $f(x)$ can be represented as a Taylor series in this interval:

$$f(x)=\sum_{n=0}^{\infty}\frac{f_n(a)}{n!}\,(x-a)^n, x\in(a-R;\ a+R).$$

24. Taylor Series for Elementary Functions

Consider the expansion of elementary functions into a power series over powers of x, which can be obtained from the series (29) for $a=0$:

$$f(x)=a(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\cdots\frac{f^{(n)}(0)}{n!}x^n, x\in(-R;+R). \tag{31}$$

The series (31) is also called the Maclaurin series.

1. $f(x)=sinx$.

$$sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^n x^{2b+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!},$$

$$x \in (-\infty; +\infty). \qquad (32)$$

2. $f(x) = cosx$.

$$cosx = x - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n x^{2b+1}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!},$$

$$x \in (-\infty; +\infty). \qquad (33)$$

It is interesting to note that the odd function *sin x* is expanded into a power series by odd powers of *x*, and the even function *cosx* is expanded by even powers of *x*.

3. $f(x) = e^x$.

$$e^x = 1 + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=\infty}^{\infty} \frac{x^n}{n!}, x \in (-\infty; +\infty).$$

4. $f(x) = \ln(1 + x)$.

$$\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} \ldots + (-1)^n \frac{x^n}{n} + \cdots = \sum_{m=1}^{\infty} (-1)^{n-1} \frac{x^n}{n},$$

$$x \in (-1; +1). \qquad (34)$$

5. $f(x) = (1 + x)^a$, where *a* is a real number.

$$(1 + x)^\alpha 1 + \frac{\alpha}{1!} x + \frac{\alpha(\alpha - 1)}{2!} x^2 + \cdots +$$

$$+ \frac{\alpha(\alpha - 1)(\alpha - 2) \ldots (\alpha - n + 1)}{n!} x^n +$$

$$+ \cdots = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)\ldots(\alpha-n+1)}{n!} x^n, \quad x \in (-1; +1). \qquad (35)$$

If $a = m$, where m is a natural number, then the function $(1 + x)^m$ is a polynomial degree m and therefore all derivatives, starting from (m+1)*th* derivatives, are equal to zero, when turning to zero and the corresponding coefficients of the series (35). From equation (35) we obtain the known expansion of the binomial $(1 + x)^m$, which is called the Newtons binomial:

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots + x^m.$$

The resulting power series can be used when finding power series for other functions.

FOURIER SERIES

25. Fluctuations and Periodic Processes. Periodic Functions

Processes repeated at certain intervals are common in nature and technology. Such processes are called periodic. Such, for example, are the oscillatory and rotational motions of various parts of machines and devices, movements of celestial bodies and elementary particles, acoustic and electromagnetic oscillations, and so on.

Periodic processes are modeled using periodic functions. The function $y = f(x)$ is said to be periodic with a period $T > 0$ if it is defined on the whole numerical axis and the equality $f(x + T) = f(x)$, $x \in R$ holds for it.

Simplest harmonic motion is oscillatory motion of a material particle, which is described by a function

$$y(x) = a\sin(kx + x_0), x \geq 0,$$

where a is the amplitude of oscillation; k is the cyclic frequency; and x_0 – initial phase.

The function $y(x)$ (and its graph) is said to be simple harmonic. By superimposing simple harmonic motion, you can get a variety of periodic oscillations.

Naturally, the inverse problem arises of the possibility of representing a periodic motion given by some periodic function as a sum of simple harmonic motion. It turned out that this, cannot be done if we limit ourselves to a finite sum of simple harmonic motion. If we enter infinite sums of simple harmonic motion, then almost every periodic function can be decomposed into simple harmonic motion.

Let $f(x)$ be a 2π periodic function. We must represent this function through the sum of the form

$$\frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots +$$
$$+a_n \cos nx + b_n \sin nx + \cdots = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx +$$
$$+b_n \sin nx\,,$$

where $a_0, a_1, b_1, \ldots, a_n, b_n, \ldots$ – some steels. The free member is taken in the form of $\frac{a_0}{2}$ for convenience.

The resulting series is called trigonometric and real numbers $a_0, a_n, b_n, (n = 1, 2, \ldots)$ – its coefficients.

26. Fourier Series of a Triangle Functions

Let $y = f(x) - 2\pi$ be a periodic function, integrated on the interval $[-\pi, \pi]$. Consider the functions of the sequence

$1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx, \ldots$

Theorem. Trigonometric system of functions 1, $\cos x$, $\sin x$, …, $\cos nx$, $\sin nx$, … is orthogonal on the segment $[-\pi, \pi]$.

Definition. A trigonometric series, the coefficients of which are the Fourier coefficients of the function $f(x)$, is called the Fourier series of this function and is written as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + +b_n \sin nx = S(x).$$

The symbol for correspondence "~" means that the Fourier series integrated on the segment $[-\pi, \pi]$ of the function $f(x)$ is aligned.

Numbers a_0, a_n and b_n, which are determined by formulas

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx, \quad n = 1,2,\dots,$$

are called Fourier coefficients of the function $f(x)$.

Lipschitz's theorem (sufficient conditions for the Fourier series representation of a function). If the function $f(x) - 2\pi$ is a periodic and piecewise-differentiated function on the interval $[-\pi, \pi]$, then the Fourier series of this function is convergent on the segment $[-\pi, \pi]$ to the function $S(x)$, and

a) $S(x) = f(x)$ – at the points of continuity of the function $f(x)$;

b) $S(x_0) = \frac{f(x_0-0)+f(x_0+0)}{2}$ – at the point x_0 of the rupture of the function;

c) $S(-\pi) = S(\pi)\frac{f(-\pi+0)+f(\pi+0)}{2}$ – at the ends of the segment.

27. Fourier Series for Even and Odd Functions

Let the function $f(x)$ be given on the interval $[-\pi, \pi]$ next to Fourier. We show that the calculation of the coefficients of this series is simplified if the function $f(x)$ is either even or odd.

Suppose, for example, that $f(x)$ is even on the interval $[-\pi, \pi]$, then $f(x) \sin nx$ (n = 1, 2, ...) is odd, and $f(x) \cos nx$, $n = 0, 1, 2, ...$ – even functions on this segment.

So,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx = 0,$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx,$$

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x) \cos nx \, dx, \qquad n = 1, 2, ...$$

by the properties of definite integrals.

Thus, the Fourier series will have the form

$$f(x) = \frac{a_0}{2}\sum_{n=1}^{\infty} a_n \cos nx.$$

Similarly, if the function $f(x)$ is odd, then its Fourier series has the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx,$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx.$$

Note that the Fourier series reflects the nature of the function. An even function is decomposed by cosines (even functions), and an odd function is decomposed by sines (odd functions).

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